

Ghodrat Ebadi · E. V. Krishnan · Stephen Johnson · Anjan Biswas

Cnoidal wave, snoidal wave, and soliton solutions of the $D(m,n)$ equation

Received: 7 June 2011 / Accepted: 15 October 2012 / Published online: 8 November 2012
© The Author(s) 2012. This article is published with open access at Springerlink.com

Abstract This paper studies the $D(m,n)$ equation, which is the generalized version of the Drinfeld–Sokolov equation. The traveling wave hypothesis and exp-function method are applied to integrate this equation. The mapping method and the Weierstrass elliptic function method also display an additional set of solutions. The kink, soliton, shock waves, singular soliton solution, cnoidal and snoidal wave solutions are all obtained by these varieties of integration tools.

Mathematics Subject Classification 37K10 · 35Q51 · 35Q55

المخلص

تقوم هذه الورقة بدراسة معادلة $D(m,n)$ والتي تمثل تعميماً لمعادلة درينفيلد–سكولوف. وقد استعملت فرضية الأمواج المسافرة وطريقة الدوال الأسية لمكاملة هذه المعادلة. كما أن طريقتي الراسم ودوال وايرستراس الإهليلجية تعرض أيضاً حلولاً إضافية. فحلول الشبك، والسيلتون، وأمواج الصدمة، وحلول السيلتون الشاذة، وأمواج snoidal نحصل عليها كلها من خلال هذه الوسائل المتنوعة لمكاملة المعادلات.

1 Introduction

The theory of nonlinear evolution equations (NLEEs) has made a lot of advances in the past few decades [1–13]. These advances turned out to be a blessing in theoretical physics and engineering sciences where these NLEEs appear on a daily basis. They appear in the study of shallow water waves in beaches and lake shores, nonlinear fiber optic communications, Langmuir and Alfvén waves in plasmas, and chiral solitons in nuclear physics, just to name a few. These NLEEs that govern the wide applications in all walks of life have one common feature, namely a soliton solution is available for these equations.

G. Ebadi
Faculty of Mathematical Sciences, University of Tabriz, Tabriz 51666-14766, Iran

E. V. Krishnan
Department of Mathematics and Statistics, Sultan Qaboos University, Muscat,
P.O.Box 36, Al-Khod 123, Sultanate of Oman

S. Johnson
Lake Forest High School, 5407 Killens Pond Road, Felton, DE 19943, USA

S. Johnson · A. Biswas (✉)
Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
E-mail: biswas.anjan@gmail.com



One of the NLEEs that is very commonly studied is the Korteweg–de Vries (KdV) equation, which is studied in the context of shallow water waves on lakes and canals. This equation was generalized to the $K(m, n)$ equation about a decade ago. The soliton solutions as well as the compacton solutions, which are solitons with compact support, were obtained. The alternate model to describe the dynamics of shallow water waves is the Boussinesq equation, which was also generalized to the $B(m, n)$ equation. The soliton solutions and other aspects of this equation were subsequently studied. The third NLEE that is studied is the Drinfeld–Sokolov (DS) equation which appears often in the context of untwisted affine Lie algebra. This paper will shine light on the generalized version of the DS equation that is known as the $D(m, n)$ equation. The integrability aspect of this equation will be the main focus of this work.

The most important part of the history of the $D(m, n)$ equation is when it was solved earlier by the ansatz method where a soliton solution was obtained [3]. This equation was then studied with generalized evolution in the same paper during 2011 [3]. During the same year, the semi-inverse variational principle was applied to obtain the solitary wave solution analytically to the DS equation [13]. In addition, the bifurcation analysis of the traveling wave solutions to the $D(m, n)$ equation was addressed in 2010 [12]. Besides these, as far as it is known, there were no further studies that were carried out with the $D(m, n)$ equation.

This paper reports the research results of the $D(m, n)$ equation that are obtained by the aid of various mathematical techniques. The traveling wave hypothesis will first be applied to obtain the solitary wave solution to the $D(m, n)$ equation for arbitrary m and n . There are a few constraint conditions that will fall out during the course of derivation of the soliton solution. Subsequently, the exp-function method will be applied in order to extract a few additional solutions. Finally, the mapping method and the Weierstrass elliptic function method will reveal several other solutions that are in terms of Weierstrass elliptic function, cnoidal waves, and snoidal waves. The limiting cases of these solutions, namely the topological solitons, also known as kinks or shock waves, and singular solitons will be given.

2 Traveling wave hypothesis

The dimensionless form of the $D(m, n)$ equation is given by

$$q_t + k(r^m)_x = 0 \quad (1)$$

$$r_t + a(r^n)_{xxx} + bq_xr + cqr_x = 0 \quad (2)$$

where a, b, c, k, m , and n are all real valued constants and the dependent variables are $q(x, t)$ and $r(x, t)$. Taking n to be 1, the $D(m, 1)$ equation is

$$\begin{aligned} q_t + k(r^m)_x &= 0 \\ r_t + ar_{xxx} + bq_xr + cqr_x &= 0 \end{aligned}$$

Using the traveling wave assumption that

$$\begin{aligned} q(x, t) &= g(s) \\ r(x, t) &= h(s) \end{aligned}$$

where

$$s = x - vt, \quad (3)$$

(1) and (2) reduce to the coupled system of ordinary differential equations as

$$-vg' + k(h^m)' = 0 \quad (4)$$

$$-vh' + ah''' + bhg' + cgh' = 0 \quad (5)$$

Integrating (4) and taking the constant of integration to be zero, since the search is for soliton solutions, leads to

$$g = \frac{k}{v}h^m \quad (6)$$



Inserting (6) into (5) yields

$$vh' = ah''' + \frac{k}{v}(mb + c)h^m h' \quad (7)$$

Integrating (7) and again taking the constant of integration to be zero, since the search in this section is for soliton solutions, gives

$$vh = ah'' + \frac{k(mb + c)}{v(m + 1)}h^{m+1} \quad (8)$$

Multiplying (8) by h' and a third time taking the constant of integration to be zero yields

$$a(h')^2 = \frac{h^2}{v} \left[v^2 - \frac{2k(mb + c)}{(m + 1)(m + 2)}h^m \right]. \quad (9)$$

After separating variables and integrating, (9) leads to

$$-2 \tanh^{-1} \left(\sqrt{1 + \frac{2k(mb + c)}{v^2(m + 1)(m + 2)}h^m} \right) = ms \sqrt{\frac{v}{a}}. \quad (10)$$

Solving (10) for $h(s)$ and using (3) gives the exact traveling wave solution to (1) and (2) as

$$q(x, t) = \frac{v(m + 1)(m + 2)}{2(mb + c)} \operatorname{sech}^2 \left[\frac{m}{2} \sqrt{\frac{v}{a}}(x - vt) \right] \quad (11)$$

and

$$r(x, t) = \left\{ \frac{v^2(m + 1)(m + 2)}{2k(mb + c)} \right\}^{\frac{1}{m}} \operatorname{sech}^{\frac{2}{m}} \left[\frac{m}{2} \sqrt{\frac{v}{a}}(x - vt) \right] \quad (12)$$

respectively. These soliton solutions (11) and (12) can be re-written as

$$q(x, t) = A_1 \operatorname{sech}^2 [B(x - vt)] \quad (13)$$

and

$$r(x, t) = A_2 \operatorname{sech}^{\frac{2}{m}} [B(x - vt)] \quad (14)$$

where the amplitudes (A_1 and A_2) of the solitary waves in (13) and (14) are given by

$$A_1 = \frac{v(m + 1)(m + 2)}{2(mb + c)} \quad (15)$$

and

$$A_2 = \left\{ \frac{v^2(m + 1)(m + 2)}{2k(mb + c)} \right\}^{\frac{1}{m}}, \quad (16)$$

while the inverse width (B) of the soliton is given by

$$B = \frac{m}{2} \sqrt{\frac{v}{a}}. \quad (17)$$

Thus Relation (17) implies that solitary waves will exist provided

$$av > 0$$

and (15) requires

$$mb + c \neq 0$$

for solitary waves to exist, while (16) shows that the solitary waves will exist provided

$$kv^2(mb + c) > 0$$

for m an even number, while for m an odd number, there is no such restriction.



3 Exponential function method

The exp-function method was first proposed by He and Wu and systematically studied for solving a class of nonlinear partial differential equations. We consider the general nonlinear partial differential equation of the type

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxx}, \dots) = 0. \quad (18)$$

Using the transformation

$$\eta = kx + wt \quad (19)$$

where k and w in (19) are constants, we can rewrite Eq. (18) as the following nonlinear ODE:

$$Q(u, u', u'', u''', \dots) = 0. \quad (20)$$

According to the exp-function method, as developed by He and Wu, we assume that the wave solutions can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n=-c}^d a_n \exp(n\eta)}{\sum_{m=-p}^q b_m \exp(m\eta)} \quad (21)$$

where p, q, d and c are positive integers which are known to be further determined, and a_n and b_m are unknown constants. We can rewrite Eq. (20) in the following equivalent form:

$$u(\eta) = \frac{a_c \exp(c\eta) + \dots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \dots + a_{-q} \exp(-q\eta)}$$

This equivalent formulation plays an important and fundamental part in finding the analytic solution of problems. To determine the values of c and p , we balance the linear term of highest order of Eq. (21) with the highest order nonlinear term. Similarly, to determine the values of d and q , we balance the linear term of lowest order of Eq. (20) with the lowest order nonlinear term.

3.1 Application to $D(m, n)$ equation

The dimensionless form of the $D(m, n)$ equation is given by

$$q_t + k(r^m)_x = 0$$

$$r_t + a(r^n)_{xxx} + bq_x r + cqr_x = 0$$

where a, b, c, k, m , and n are all real constants and the dependent variables are $q(x, t)$ and $r(x, t)$. In particular, m and n are natural numbers. Taking n to be 1, the $D(m, 1)$ equation is

$$q_t + k(r^m)_x = 0 \quad (22)$$

$$r_t + ar_{xxx} + bq_x r + cqr_x = 0 \quad (23)$$

Using the traveling wave assumption

$$q(x, t) = g(s)$$

$$r(x, t) = h(s)$$

where

$$s = x - vt,$$

(22) and (23) reduce to the coupled system of ordinary differential equations as

$$-vg' + k(h^m)' = 0 \quad (24)$$

$$-vh' + ah''' + bhg' + cgh' = 0 \quad (25)$$



Integrating (24) and taking the constant of integration to be zero, since the search is for soliton solutions, leads to

$$g = \frac{k}{v} h^m \quad (26)$$

Inserting (26) into (25) yields

$$vh' = ah''' + \frac{k}{v}(mb + c)h^m h' \quad (27)$$

Integrating (27) and again taking the constant of integration to be zero, since the search in this section is for soliton solutions, gives

$$vh = ah'' + \frac{k(mb + c)}{v(m + 1)} h^{m+1} \quad (28)$$

Multiplying (28) by h' and a third time taking the constant of integration to be zero yields

$$a(h')^2 = \frac{h^2}{v} \left[v^2 - \frac{2k(mb + c)}{(m + 1)(m + 2)} h^m \right]. \quad (29)$$

Introducing the transformation

$$h = W^{\frac{2}{m}}$$

Equation (29) becomes

$$(m + 2)(m + 1)(-v^2 m^2 W^2 + 4av(W')^2) + 2km^2(bm + c)W^4 = 0 \quad (30)$$

According to the Exp-function method, we assume that the solution of Eq. (30) can be expressed in the form

$$W(s) = \frac{a_c \exp(cs) + \dots + a_{-d} \exp(-ds)}{b_p \exp(ps) + \dots + b_{-q} \exp(-qs)} \quad (31)$$

By balancing the highest order of linear term $(W')^2$ with the highest order nonlinear term W^4 we get

$$W^4(s) = \frac{c_1 \exp(4cs) + \dots}{c_2 \exp(4ps) + \dots}$$

and

$$(W')^2 = \frac{c_3 \exp((2c + 2p)s) + \dots}{c_4 \exp(4ps) + \dots} \quad (32)$$

we have $4p = 2c + 2p$, which leads to the result $p = c$. Similarly, balancing the lowest order of linear term $(W')^2$ with the highest order nonlinear term W^4 we obtain

$$W^4(s) = \frac{c_5 \exp(-4ds) + \dots}{c_6 \exp(-4qs) + \dots} \quad (33)$$

and

$$(W')^2 = \frac{c_7 \exp(-(2d + 2q)s) + \dots}{c_8 \exp(-4qs) + \dots}.$$

Therefore, we can obtain the following relation $-4d = -2d - 2q$, resulting in $d = q$. We can freely choose the values of c and d , but the final solution does not strongly depend upon the choice of values of c and d . For simplicity, we set $p = c = 1$ and $d = q = 1$, then Eq. (31) becomes

$$W(s) = \frac{a_{-1} \exp(-s) + a_0 + a_1 \exp(s)}{b_{-1} \exp(-s) + b_0 + b_1 \exp(s)} \quad (34)$$



Substituting Eqs. (32), (33), and (34) into Eq. (30), and equating to zero the coefficients of all powers of $\exp(n\xi)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}$ and v . Solving the system of algebraic equations by the help of Maple, we obtain

Case 1:

$$a_1 = 0, \quad a_{-1} = 0, \quad b_0 = 0, \quad b_1 = \frac{a_0 k m^4 (b m + c)}{32 a^2 b_{-1} (m^2 + 3m + 2)}, \quad v = \frac{4a}{m^2}. \quad (35)$$

Substituting Eq. (35) into Eq. (34) yields

$$W(s) = \frac{a_0}{b_{-1} e^{-s} + \frac{a_0 k m^4 (b m + c)}{32 a^2 b_{-1} (m^2 + 3m + 2)} e^s},$$

where $s = x - vt$. Thus, the exact solution to (22) and (23) is

$$q(x, t) = \frac{k}{v} \left\{ \frac{a_0}{b_{-1} e^{-s} + \frac{a_0 k m^4 (b m + c)}{32 a^2 b_{-1} (m^2 + 3m + 2)} e^s} \right\}^2,$$

$$r(x, t) = \left\{ \frac{a_0}{b_{-1} e^{-s} + \frac{a_0 k m^4 (b m + c)}{32 a^2 b_{-1} (m^2 + 3m + 2)} e^s} \right\}^{\frac{2}{m}},$$

where a_0 and b_{-1} are arbitrary constants.

Case 2:

$$a_1 = 0, \quad b_1 = 0, \quad b_0 = \frac{z a_0}{v}, \quad b_{-1} = \frac{2 a_{-1} k (m b + c)}{z v (m + 1) (m + 2)}. \quad (36)$$

Substituting Eq. (36) into Eq. (34) yields

$$W(s) = \frac{a_0 + a_{-1} e^{-s}}{\frac{z a_0}{v} + \frac{2 a_{-1} k (m b + c)}{z v (m + 1) (m + 2)} e^{-s}},$$

where $s = x - vt$. Thus, the exact solution to (22) and (23) is

$$q(x, t) = \frac{k}{v} \left\{ \frac{a_0 + a_{-1} e^{-s}}{\frac{z a_0}{v} + \frac{2 a_{-1} k (m b + c)}{z v (m + 1) (m + 2)} e^{-s}} \right\}^2,$$

$$r(x, t) = \left\{ \frac{a_0 + a_{-1} e^{-s}}{\frac{z a_0}{v} + \frac{2 a_{-1} k (m b + c)}{z v (m + 1) (m + 2)} e^{-s}} \right\}^{\frac{2}{m}},$$

where z is the root of the following equation.

$$(m^2 + 3m + 2)z^2 - 2ck - 2mb = 0.$$

and a_0 and a_{-1} are arbitrary constants.

Case 3:

$$a_0 = a_1 = b_0 = b_1 = 0, \quad v = \frac{16a}{m^2}. \quad (37)$$

Substituting Eq. (37) into Eq. (34) yields

$$W(s) = \frac{a_{-1} e^{-s}}{b_1 e^s},$$



where $s = x - vt$. Thus, the exact solution to (22) and (23) is

$$q(x, t) = \frac{k}{v} \left(\frac{a_{-1} e^{-s}}{b_1 e^s} \right)^2,$$

$$r(x, t) = \left(\frac{a_{-1} e^{-s}}{b_1 e^s} \right)^{\frac{2}{m}},$$

where a_{-1} and b_1 are arbitrary constants.

Case 4:

$$a_0 = b_1 = b_{-1} = 0, \quad v = \frac{-28a}{m^2}, \quad a_1 = \frac{112a^2 b_0^2 (2 + 3m + m^2)}{m^4 a_{-1} k (c + bm)}. \quad (38)$$

Substituting Eq. (38) into Eq. (34) yields

$$W(s) = \frac{a_{-1} e^{-s} + \frac{112a^2 b_0^2 (2 + 3m + m^2)}{m^4 a_{-1} k (c + bm)} e^s}{b_0},$$

where $s = x - vt$. Thus, the exact solution to (22) and (23) is

$$q(x, t) = \frac{k}{v} \left[\frac{a_{-1} e^{-s} + \frac{112a^2 b_0^2 (2 + 3m + m^2)}{m^4 a_{-1} k (c + bm)} e^s}{b_0} \right]^2,$$

$$r(x, t) = \left[\frac{a_{-1} e^{-s} + \frac{112a^2 b_0^2 (2 + 3m + m^2)}{m^4 a_{-1} k (c + bm)} e^s}{b_0} \right]^{\frac{2}{m}},$$

where a_{-1} and b_0 are arbitrary constants. The results of this section are all in terms of exponential functions and are all reducible to soliton solutions or pure exponential functions by a proper choice of the arbitrary constants that appear.

4 Mapping methods and Weierstrass elliptic function method

Now, we solve the coupled Eqs. (1) and (2) for $m = 1$, $n = 1$ by mapping methods and Weierstrass elliptic function (WEF) method [5–8] which generate a variety of periodic wave solutions (PWSs) in terms of squared Jacobi elliptic functions (JEFs) and we subsequently derive their infinite period counterparts in terms of hyperbolic functions which are solitary wave solutions (SWSs), shock wave solutions or singular solutions.

When $m = 1$, $n = 1$, the traveling wave solutions (TWSs) in the form of Eq. (3) reduce Eqs. (1) and (2) to

$$Ar'' + Br + Cr^2 = 0, \quad (39)$$

where

$$A = a, \quad B = cq - v, \quad C = \frac{bk}{2v}.$$

and prime indicates differentiation with respect to s .



4.1 Mapping method

Here, we assume that Eq. (39) has a solution in the form

$$r = A_0 + A_1 f, \quad (40)$$

where

$$f'' = P + Qf + Rf^2, \quad f'^2 = 2Pf + Qf^2 + \frac{2}{3}Rf^3. \quad (41)$$

Equation (40) is the mapping relation between the solution to Eq. (41) and that of Eq. (39).

We substitute Eq. (40) into Eq. (39), use Eq. (41) and equate the coefficients of like powers of f to zero so that we arrive at the set of equations

$$AA_1R + CA_1^2 = 0, \quad (42)$$

$$AA_1Q + BA_1 + 2CA_0A_1 = 0, \quad (43)$$

$$AA_1P + BA_0 + 2CA_0^2 = 0. \quad (44)$$

From Eqs. (42) and (43), we obtain $A_1 = -\frac{AR}{C}$ and $A_0 = -\frac{AQ+B}{2C}$. Equation (44) gives rise to the constraint relation

$$A^2(Q^2 - 4PR) = B^2. \quad (45)$$

Case 1. $P = 2$, $Q = -4(1 + m^2)$, $R = 6m^2$.

Now Eq. (41) has two solutions $f(s) = \text{sn}^2(s)$ and $f(s) = \text{cd}^2(s)$.

Therefore, we obtain the PWSs of Eqs. (1) and (2) as

$$r(x, t) = \frac{v[v + 4a(1 + m^2) - cq]}{bk} - \frac{12avm^2}{bk} \text{sn}^2(x - vt), \quad (46)$$

and

$$r(x, t) = \frac{v[v + 4a(1 + m^2) - cq]}{bk} - \frac{12avm^2}{bk} \text{cd}^2(x - vt).$$

As $m \rightarrow 1$, Eq. (46) gives rise to the SWS

$$r(x, t) = \frac{v(v + 8a - cq)}{bk} - \frac{12av}{bk} \tanh^2(x - vt). \quad (47)$$

Case 2. $P = 2(1 - m^2)$, $Q = 4(2m^2 - 1)$, $R = -6m^2$.

Now Eq. (41) has the solution $f(s) = \text{cn}^2(s)$.

Therefore, the PWS of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2m^2 - 1) - cq]}{bk} + \frac{12avm^2}{bk} \text{cn}^2(x - vt). \quad (48)$$

As $m \rightarrow 1$, Eq. (48) will lead us to the same SWS (47).

Case 3. $P = -2(1 - m^2)$, $Q = 4(2 - m^2)$, $R = -6$.

Now Eq. (41) has the solution $f(s) = \text{dn}^2(s)$.

Therefore, the PWS of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2 - m^2) - cq]}{bk} + \frac{12av}{bk} \text{dn}^2(x - vt). \quad (49)$$

As $m \rightarrow 1$ in Eq. (49), the same SWS (47) will be retained.



Case 4. $P = 2m^2$, $Q = -4(1 + m^2)$, $R = 6$.

Now Eq. (41) has the solutions $f(s) = ns^2(s)$ and $f(s) = dc^2(s)$.

Therefore, the PWSs of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v + 4a(1 + m^2) - cq]}{bk} - \frac{12av}{bk}ns^2(x - vt),$$

and

$$r(x, t) = \frac{v[v + 4a(1 + m^2) - cq]}{bk} - \frac{12av}{bk}dc^2(x - vt). \quad (50)$$

As $m \rightarrow 1$, Eq. (50) will give rise to the singular solution

$$r(x, t) = \frac{v(v + 8a - cq)}{bk} - \frac{12av}{bk}\coth^2(x - vt). \quad (51)$$

Case 5. $P = -2m^2$, $Q = 4(2m^2 - 1)$, $R = 6(1 - m^2)$.

Here, Eq. (41) has the solution $f(s) = nc^2(s)$.

So, the PWSs of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2m^2 - 1) - cq]}{bk} - \frac{12av(1 - m^2)}{bk}nc^2(x - vt). \quad (52)$$

Case 6. $P = -2$, $Q = 4(2 - m^2)$, $R = -6(1 - m^2)$.

In this case, Eq. (41) has the solution $f(s) = nd^2(s)$.

Therefore, the PWS of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2 - m^2) - cq]}{bk} + \frac{12av(1 - m^2)}{bk}nd^2(x - vt).$$

Case 7. $P = 2$, $Q = 4(2 - m^2)$, $R = 6(1 - m^2)$.

Therefore, Eq. (41) has the solution $f(s) = sc^2(s)$.

Thus the PWS of Eqs. (1) and (2) is the same as the solution (52).

Case 8. $P = 2$, $Q = 4(2m^2 - 1)$, $R = -6m^2(1 - m^2)$.

Here, Eq. (41) has the solution $f(s) = sd^2(s)$.

Therefore, the PWS of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2m^2 - 1) - cq]}{bk} + \frac{12avm^2(1 - m^2)}{bk}sd^2(x - vt).$$

Case 9. $P = 2(1 - m^2)$, $Q = 4(2 - m^2)$, $R = 6$.

Therefore, Eq. (41) has the solution $f(s) = cs^2(s)$.

Therefore, the PWS of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2 - m^2) - cq]}{bk} - \frac{12av}{bk}cs^2(x - vt). \quad (53)$$

As $m \rightarrow 1$, Eq. (53) will give rise to the singular solution (51).

Case 10. $P = -2m^2(1 - m^2)$, $Q = 4(2m^2 - 1)$, $R = 6$.

Thus, Eq. (41) has the solution $f(s) = ds^2(s)$.

Here, the PWS of Eqs. (1) and (2) will be

$$r(x, t) = \frac{v[v - 4a(2m^2 - 1) - cq]}{bk} - \frac{12av}{bk}ds^2(x - vt). \quad (54)$$

As $m \rightarrow 1$, Eq. (54) will give rise to the same singular solution (51).

It is evident from the constraint relation (45) that $Q^2 - 4PR$ should always be positive with our choices of P , Q and R for real solutions to exist. In all the cases considered, we can see that $Q^2 - 4PR$ is equal to $16m^4 - 16m^2 + 16$ which is always positive for $0 < m < 1$. Thus, all our solutions are valid with the constraint relation (45).



The solutions derived in this section are all in terms of doubly periodic functions and are typically referred to as cnoidal waves, snoidal waves or dnoidal waves.

4.2 Modified mapping method

In this case, we assume that Eq. (39) has a solution in the form

$$r = A_0 + A_1 f + B_1 f^{-1}, \quad (55)$$

where f satisfies Eq. (41).

Equation (55) is the mapping relation between the solution to Eq. (41) and that of Eq. (39).

We substitute Eq. (55) into Eq. (39), use Eq. (41) and equate the coefficients of like powers of f to zero so that we will obtain a set of equations giving rise to the solutions

$$A_0 = -\frac{AQ + B}{2C}, \quad A_1 = -\frac{AR}{C}, \quad B_1 = -\frac{3PA}{C},$$

and the constraint relation

$$A^2(Q^2 + 16PR) = B^2. \quad (56)$$

Case 1. $P = 2$, $Q = -4(1 + m^2)$, $R = 6m^2$.

Now, Eq. (41) has two solutions $f(s) = \text{sn}^2(s)$ and $f(s) = \text{cd}^2(s)$.

Thus the PWSs of Eqs. (1) and (2) are

$$r(x, t) = \frac{v[v + 4a(1 + m^2) - cq]}{bk} - \frac{12av}{bk} (m^2 \text{sn}^2(x - vt) + n \text{sn}^2(x - vt)) \quad (57)$$

and

$$r(x, t) = \frac{v[v + 4a(1 + m^2) - cq]}{bk} - \frac{12av}{bk} (m^2 \text{cd}^2(x - vt) + d \text{cd}^2(x - vt)).$$

When $m \rightarrow 1$, Eq. (57) will degenerate to the solution

$$r(x, t) = \frac{v(v + 8a - cq)}{bk} - \frac{12av}{bk} (\tanh^2(x - vt) + \coth^2(x - vt)). \quad (58)$$

Equation (58) represents a combination of a topological soliton and a singular soliton.

Case 2. $P = -2(1 - m^2)$, $Q = 4(2 - m^2)$, $R = -6$.

Therefore, Eq. (41) has the solution $f(s) = \text{dn}^2(s)$.

In this case, the PWS of Eqs. (1) and (2) is

$$r(x, t) = \frac{v[v - 4a(2 - m^2) - cq]}{bk} + \frac{12av}{bk} (\text{dn}^2(x - vt) + (1 - m^2)n \text{dn}^2(x - vt)). \quad (59)$$

When $m \rightarrow 1$, Eq. (59) will give rise to the SWS (47).

Case 3. $P = 2$, $Q = 4(2 - m^2)$, $R = 6(1 - m^2)$.

Here, Eq. (41) has the solution $f(s) = \text{sc}^2(s)$.

Now, the PWS of Eqs. (1) and (2) is

$$r(x, t) = \frac{v[v - 4a(2 - m^2) - cq]}{bk} - \frac{12av}{bk} ((1 - m^2) \text{sc}^2(x - vt) + c \text{sc}^2(x - vt)). \quad (60)$$

When $m \rightarrow 1$, Eq. (60) will lead to the singular solution (51).



The expression $Q^2 + 16PR$ reduces to $16m^4 + 224m^2 + 16$ in case 1 and to $16m^4 - 256m^2 + 256$ in cases 2 and 3. Both of them are positive for $0 \leq m \leq 1$. Thus all our solutions are valid with the constraint relation (56).

4.3 Weierstrass elliptic function method

Here, we will present the solutions in terms of WEFs. For this purpose, we multiply Eq. (39) by r and rewrite it as

$$Arr'' + Br^2 + Cr^3 = 0. \quad (61)$$

We assume that

$$r(s) = \lambda \wp(s) + \mu \quad (62)$$

is a solution to Eq. (61), where λ and μ are constants to be determined. Here, $\wp(s)$ is the WEF which satisfies the equation

$$\wp'^2(s) = 4\wp^3(s) - g_2\wp(s) - g_3,$$

where g_2 and g_3 are the invariants of the WEF which satisfies the inequality

$$g_2^3 - 27g_3^2 > 0. \quad (63)$$

Substituting Eq. (62) into Eq. (61), and equating the coefficients of powers of $\wp(s)$ to zero, we obtain

$$6A\lambda^2 + C\lambda^3 = 0, \quad (64)$$

$$6A\lambda\mu + B\lambda^2 + 3C\lambda^2\mu = 0, \quad (65)$$

$$-\frac{1}{2}A\lambda^2g_2 + 2\lambda B\mu + 3C\lambda\mu^2 = 0, \quad (66)$$

$$-\frac{1}{2}A\lambda\mu g_2 + B\mu^2 + 3C\mu^3 = 0. \quad (67)$$

From Eqs. (64) and (65), we get $\lambda = -\frac{6A}{C}$ and $\mu = -\frac{B}{2C}$ respectively.

From Eq. (66), we obtain

$$g_2 = \frac{B^2}{12A^2}. \quad (68)$$

It may be observed that g_2 has to be positive to satisfy the inequality (63) and it is indeed so from Eq. (68). Also, Eq. (67) is automatically satisfied by the values of λ , μ and g_2 .

Thus the solution of Eq. (61) is,

$$r(s) = -\frac{12av}{bk}\wp(s) + \frac{v(v-cq)}{bk}.$$

Now we can write the corresponding periodic wave solutions in terms of JEFs in three different forms as follows:

$$r(s) = -\frac{12av}{bk} \{e_3 + (e_1 - e_3)\text{ns}^2(\sqrt{e_1 - e_3}s, \sqrt{m})\} + \frac{v(v-cq)}{bk}, \quad (69)$$

$$r(s) = -\frac{12av}{bk} \{e_1 + (e_1 - e_3)\text{cs}^2(\sqrt{e_1 - e_3}s, \sqrt{m})\} + \frac{v(v-cq)}{bk}, \quad (70)$$

$$r(s) = -\frac{12av}{bk} \{e_2 + (e_1 - e_3)\text{ds}^2(\sqrt{e_1 - e_3}s, \sqrt{m})\} + \frac{v(v-cq)}{bk}. \quad (71)$$

Here, e_1, e_2, e_3 satisfy the cubic equation

$$4z^3 - g_2z - g_3 = 0 \quad (72)$$



with

$$e_1 > e_2 > e_3$$

and m , the modulus of the JEF is given by

$$m = \frac{e_2 - e_3}{e_1 - e_3}.$$

In the infinite period limit, as $m \rightarrow 1$, $e_1 \rightarrow e_2$ and $\text{sn}(s) \rightarrow \tanh(s)$, $\text{cn}(s) \rightarrow \text{sech}(s)$, $\text{dn}(s) \rightarrow \text{sech}(s)$. From the cubic equation (72), we have $e_1 + e_2 + e_3 = 0$, which leads to $e_3 = -2e_1$.

Hence we can deduce the SWS from Eq. (69) as

$$r(x, t) = -\frac{12av}{bk} \frac{e_1 + 2e_1 \text{sech}^2(\sqrt{3e_1}(x - vt))}{1 - \text{sech}^2(\sqrt{3e_1}(x - vt))} + \frac{v(v - cq)}{bk},$$

and the singular soliton solution from Eqs. (70) and (71) as

$$r(x, t) = -\frac{12av}{bk} \left[e_1 + 3e_1 \text{csch}^2(\sqrt{3e_1}(x - vt)) \right] + \frac{v(v - cq)}{bk}.$$

5 Conclusions

This paper addressed the $D(m, n)$ equation by the aid of a few integration tools. These techniques of integration displayed a plethora of solutions to the equation. The traveling wave hypothesis obtained the soliton solutions along with a few constraints that must hold in order for the solitons to exist. The exp-function approach also yielded a list of several other forms of solutions to the $D(m, n)$ equation. These solutions are several mixtures and rational combinations of the hyperbolic functions, although they are written in terms of exponential functions. The special values of the coefficients of these exponential functions will give the hyperbolic functions. Subsequently, the mapping method lead to the cnoidal and snoidal wave solutions to the equation of study. These solutions in the limiting cases were periodic solutions, singular periodic solutions, kinks or shock waves, singular solitons or solitary waves.

This paper therefore listed a profound and stupendous list of results that are being reported for the first time in the context of this equation. In the future, there are several other aspects of this equation that will be touched upon. These include the stochasticity aspects of the equation. For example, the $D(m, n)$ equation will be studied when one or more of the coefficients are random, rather than being deterministic. Soliton perturbation theory will also be addressed where perturbation terms will be taken into account. The conservation laws will also be determined to this equation. Additionally, the numerical integration technique will be applied in order to address this equation. These form only the tip of the iceberg.

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Afrouzi, G.A.; Vahidi, J.; Saeidy, M.: Numerical solutions of generalized Drinfeld-Sokolov equations using the homotopy analysis method. *Int. J. Nonlinear Sci.* **9**(2), 165–170 (2010)
2. Bekir, A.: New exact traveling wave solutions for regularized long-wave, Phi-Four and Drinfeld-Sokolov equations. *Int. J. Nonlinear Sci.* **6**(1), 46–52 (2008)
3. Biswas, A.; Triki, H.: 1-soliton solution of the $D(m, n)$ equation with generalized evolution. *Appl. Math. Comput.* **217**(21), 8490–8496 (2011)
4. Kara, A.H.: An analysis of the symmetries and conservation laws of the class of Zakharov–Kuznetsov equations. *Math. Comput. Appl.* **15**(4), 658–664 (2010)
5. Krishnan, E.V.; Biswas, A.: Solutions of the Zakharov–Kuznetsov equation with higher order nonlinearity by mapping and ansatz method. *Phys. Wave Phenom.* **18**(4), 256–261 (2010)
6. Krishnan, E.V.; Peng, Y.: A new solitary wave solution for the new Hamiltonian amplitude equation. *J. Phys. Soc. Japan.* **74**, 896–897 (2005)



7. Krishnan, E.V.; Peng, Y.: Squared Jacobian elliptic function solutions for the (2+1)-D Korteweg de Vries equation. *Adv. Theoret. Appl. Math.* **1**, 9–25 (2006)
8. Krishnan, E.V.; Peng, Y.: Exact solutions to the combined KdV–mKdV equation by the extended mapping method. *Phys. Scripta*. **73**, 405–409 (2006)
9. Wazwaz, A.M.: Exact and explicit traveling wave solutions for the nonlinear Drinfeld–Sokolov system. *Commun. Nonlinear Sci. Numer. Simul.* **11**(3), 311–325 (2006)
10. Wazwaz, A.M.: The Cole–Hopf transformation and multiple soliton solutions for the integrable sixth order Drinfeld–Sokolov–Satsuma–Hirota equation. *Appl. Math. Comput.* **207**(1), 248–255 (2009)
11. Wu, L.; Chen, S.; Pang, C.: Traveling wave solutions for generalized Drinfeld–Sokolov equations. *Appl. Math. Modell.* **33**(11), 4126–4130 (2009)
12. Zhang, K.; Tang, S.; Wang, Z.: Bifurcation of traveling wave solutions for the nonlinear dispersion Drinfel’d–Sokolov $D(m, n)$ system. *Appl. Math. Comput.* **217**(4), 1620–1631 (2010)
13. Zhang, W.-M.: Solitary solutions and singular periodic solutions of the Drinfeld–Sokolov–Wilson equation by variational approach. *Appl. Math. Sci.* **5**(38), 1887–1894 (2011)

